Quasimodular solutions of a differential equation of hypergeometric type

Masanobu Kaneko and Masao Koike

§1. Introduction and Main Theorem

In our previous paper [2], we studied further the solutions of the following differential equation in the upper half-plane $\mathfrak{H}$ which was originally found and studied in [4] in connection with the arithmetic of supersingular elliptic curves;

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0.$$

Here, k is an integer or half an integer, the symbol $'$ denotes the differentiation $(2\pi i)^{-1}d/d\tau = q\cdot d/dq$ ($q = e^{2\pi i\tau}$), and $E_2(\tau)$ is the "quasimodular" Eisenstein series of weight 2 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$:

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n.$$

Let $p \geq 5$ be a prime number and $F_{p-1}(\tau)$ be the solution of the above differential equation for $k = p-1$ which is modular on $\mathrm{SL}_2(\mathbb{Z})$ (such a solution exists and is unique up to a scalar multiple). For any zero τ_0 in $\mathfrak{H}$ of the form $F_{p-1}(\tau)$, the value of the j-function at τ_0 is algebraic and its reduction modulo (an extension of) p is a supersingular j-invariant of characteristic p, and conversely, all the supersingular j-invariants are obtained in this way from the single solution $F_{p-1}(\tau)$ with suitable choices of τ_0. This is the arithmetic connection that motivated our study of the differential equation.

Various modular forms on $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups were obtained in [2] as solutions to this differential equation, the groups depending on the choice of k. Every modular solution is expressed in terms of a hypergeometric polynomial in a suitable modular function (hence the "hypergeometric type" in the title of the paper), also depending on the choice of k. For instance, if $k \equiv 0, 4 \bmod 12$, we have a modular solution

$$E_4(\tau)^{\frac{k}{4}}F\Big(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6}; \frac{1728}{j(\tau)}\Big),$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

where

$$F(a,b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\qquad (a)_n=a(a+1)\cdots(a+n-1)$$

is the Gauss hypergeometric series (which becomes a polynomial when a or b is a negative integer, which is the case here), $E_4(\tau)$ the Eisenstein series of weight 4 on $\mathrm{SL}_2(\mathbb{Z})$, and $j(\tau)$ the elliptic modular invariant.

In addition to the modular solutions, quite remarkable was an occurrence of a *quasimodular form*, not of weight k as in the modular case but of weight $k+1$. In the present paper, we give another supply of examples of quasimodular forms as solutions to an analogous differential equation attached to the group $\Gamma_0^*(2)$, which is *not* contained in $\mathrm{SL}_2(\mathbb{Z})$;

$$\Gamma_0^*(2)=\left\langle \Gamma_0(2),\ \begin{pmatrix}0&-1\\2&0\end{pmatrix}\right\rangle$$

where

$$\Gamma_0(2)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})\,|\ c\equiv 0\mod 2\right\}.$$

($\Gamma_0^*(2)$ is the triangular group "$2A$" in the notation of Conway-Norton [1].)

Let

$$E_{2A}(\tau):=(E_2(\tau)+2E_2(2\tau))/3=1-8q-40q^2-32q^3-\cdots$$

be the quasimodular form of weight 2 on $\Gamma_0^*(2)$ which is the logarithmic derivative of the form

$$\Delta_{2A}(\tau):=\eta(\tau)^8\eta(2\tau)^8=q-8q^2+12q^3+64q^4-\cdots$$

of weight 8 on $\Gamma_0^*(2)$; $E_{2A}(\tau)=\Delta_{2A}'(\tau)/\Delta_{2A}(\tau)$, an analogous situation in the $SL_2(\mathbb{Z})$ case where $E_2(\tau)$ is the logarithmic derivative of the Ramanujan $\Delta(\tau)$. Consider the following differential equation;

$$(\#)_k\qquad f''(\tau)-\frac{k+1}{4}E_{2A}(\tau)f'(\tau)+\frac{k(k+1)}{8}E_{2A}'(\tau)f(\tau)=0.$$

Solutions which are modular on the group $\Gamma_0^*(2)$ and its subgroups were studied in [6, 7]. In particular, when k is a non-negative integer congruent to 0 or 6 modulo 8, the equation $(\#)_k$ has a one dimensional space of solutions which are modular on the group $\Gamma_0^*(2)$ itself. We note here that the equation $(\#)_k$ has a characterization by the invariance of the space of solutions under the action of $\Gamma_0^*(2)$, similar to the previous case for $\mathrm{SL}_2(\mathbb{Z})$, owing to the fact that there is no holomorphic modular form of weight 2 on $\Gamma_0^*(2)$ (see [5] and [2, §5]). By a general theory of ordinary differential equations, we see that the equation $(\#)_k$ has a quasimodular solution (which, since its transformation under $\tau\to-1/2\tau$ is also a solution, inevitably gives a solution having $\log q$ term in the expansion at $q=0$) only when k is a positive integer congruent to 3 modulo 4.

In the following, we show there indeed exists a quasimodular solution in this case and describe explicitly the solution in terms of a certain orthogonal polynomials. First we need to develop some notations. Put

$$\begin{aligned} C(\tau) &:= 2E_2(2\tau) - E_2(\tau) \\ &= 1 + 24\sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ d:\mathrm{odd}}} d\,\Big)q^n = 1 + 24q + 24q^2 + 96q^3 + \cdots, \end{aligned}$$

$$D(\tau) := \frac{\eta(2\tau)^{16}}{\eta(\tau)^8} = \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ d:\mathrm{odd}}} (n/d)^3\Big)q^n = q + 8q^2 + 28q^3 + 64q^4 + \cdots,$$

where

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}} - q^{\frac{25}{24}} - q^{\frac{49}{24}} + q^{\frac{121}{24}} + \cdots$$

is the Dedekind eta function. The functions $C(\tau)$ and $D(\tau)$ are modular forms of respective weights 2 and 4 on the group $\Gamma_0(2)$ (="$2B$") and the graded ring of modular forms of integral weights on $\Gamma_0(2)$ is generated by these $C(\tau)$ and $D(\tau)$. Recall that (see [3]) an element of degree k in the graded ring $\mathbb{C}[E_2(\tau), C(\tau), D(\tau)]$, where the generators $E_2(\tau), C(\tau), D(\tau)$ have degrees 2, 2, and 4 respectively, is referred to as a quasimodular form of weight k (on $\Gamma_0(2)$). Incidentally, the graded ring of modular forms of integral weights on $\Gamma_0^*(2)$ is generated by three elements $C(\tau)^2 = (E_4(\tau) + 4E_4(2\tau))/5$, $C(\tau)^3 - 128C(\tau)D(\tau) = (E_6(\tau) + 8E_6(2\tau))/9$, and $\Delta_{2A}(\tau)$ of respective weights 4 , 6 , 8, of which $C(\tau)^2$ and $\Delta_{2A}(\tau)$ generate freely the subring consisting forms of weight being multiple of 4, and the whole space as a graded module is generated over this ring by $C(\tau)^3 - 128C(\tau)D(\tau)$.

Now define a sequence of polynomials $P_n(x)$ $(n = 0, 1, 2, \dots)$ by

$$P_0(x) = 1,\ P_1(x) = x,\quad P_{n+1}(x) = xP_n(x) + \lambda_n P_{n-1}(x) \quad (n = 1, 2, \dots)$$

where

$$\lambda_n = 4\frac{(4n+1)(4n+3)}{n(n+1)}.$$

First few examples are

$$P_2(x) = x^2 + 70,\ P_3(x) = x^3 + 136x,\ P_4(x) = x^4 + 201x^2 + 4550, \dots.$$

The $P_n(x)$ is even or odd polynomial according as n is even or odd. We also define a second series of polynomials $Q_n(x)$ by the same recursion (with different initial values):

$$Q_0(x) = 0,\ Q_1(x) = 1,\quad Q_{n+1}(x) = xQ_n(x) + \lambda_n Q_{n-1}(x) \quad (n = 1, 2, \dots),$$

a couple of examples being

$$Q_2(x) = x,\ Q_3(x) = x^2 + 66,\ Q_4(x) = x^3 + 131x, \dots.$$

The $Q_n(x)$ has opposite parity: It is even if n is odd and odd if n is even.

Put $G(\tau) = C(\tau)^2 - 128D(\tau)$ $(= (4E_4(2\tau) - E_4(\tau))/3)$.

Theorem. *Let $k=4n+3$ $(n=0,1,2,\dots)$. The following quasimodular form of weight $k+1$ on $\Gamma_0(2)$ is a solution of $(\#)_k$:*

$$\sqrt{\Delta_{2A}(\tau)}^{\,n} P_n\Big(\frac{G(\tau)}{\sqrt{\Delta_{2A}(\tau)}}\Big)\frac{C'(\tau)}{24}-\sqrt{\Delta_{2A}(\tau)}^{\,n+1}Q_n\Big(\frac{G(\tau)}{\sqrt{\Delta_{2A}(\tau)}}\Big).$$

Remark. The appearance of the square root $\sqrt{\Delta_{2A}(\tau)}$ in the formula is superficial because of the parities of $P_n(x)$ and $Q_n(x)$, that is, the form is actually an element in $\mathbb{Q}[E_2(\tau),C(\tau),D(\tau)]$, by noting $\Delta_{2A}(\tau)=D(\tau)(C(\tau)^2-64D(\tau))$ and $C'(\tau)=(E_2(\tau)C(\tau)-C(\tau)^2)/6+32D(\tau)$. The form does not belong to $\Gamma_0^*(2)$.

§2. Proof of Theorem

It is convenient to introduce the operator ϑ_k defined by

$$\vartheta_k(f)(\tau)=f'(\tau)-\frac{k}{8}E_{2A}(\tau)f(\tau).$$

By the quasimodular property of $E_2(\tau)$ or the fact that $E_{2A}(\tau)$ is the logarithmic derivative of $\Delta_{2A}(\tau)$, we have the transformation formulas

$$E_{2A}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2E_{2A}(\tau)+\frac{4}{\pi i}c(c\tau+d)\qquad \left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(2)\right)$$

and

$$E_{2A}\left(-\frac{1}{2\tau}\right)=2\tau^2E_{2A}(\tau)+\frac{8}{\pi i}\tau.$$

From these we see that if f is modular of weight k on a subgroup of $\Gamma_0^*(2)$, then $\vartheta_k(f)$ is modular of weight $k+2$ on the same group. If f and g have weights k and l, the Leibniz rule

$$\vartheta_{k+l}(fg)=\vartheta_k(f)g+f\vartheta_l(g)$$

holds. We sometimes drop the suffix of the operator ϑ_k when the weights of modular forms we consider are clear. With this operator, the equation $(\#)_k$ can be rewritten as

$$(\#')_k\qquad \vartheta_{k+2}\vartheta_k(f)(\tau)=\frac{k(k+2)}{64}C(\tau)^2f(\tau),$$

(use $E'_{2A}(\tau)=(E_{2A}(\tau)^2-C(\tau)^2)/8$).

Denote the form in the theorem by $F_k(\tau)$. We first establish the recurrence relation (note $n=(k-3)/4$):

$$(1)\qquad F_{k+4}(\tau)=G(\tau)F_k(\tau)+\lambda_n\Delta_{2A}(\tau)F_{k-4}(\tau).$$

This is a consequence of the recursion of P_n and Q_n, namely (we often omit the variable τ hereafter)

$$\begin{aligned}
&GF_k + \lambda_n \Delta_{2A} F_{k-4} \\
&= G\Big(\sqrt{\Delta_{2A}}^{\,n} P_n\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\frac{C'}{24} - \sqrt{q\Delta_{2A}}^{\,n+1} Q_n\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\Big) \\
&+ \lambda_n \Delta_{2A}\Big(\sqrt{\Delta_{2A}}^{\,n-1} P_{n-1}\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\frac{C'}{24} - \sqrt{\Delta_{2A}}^{\,n} Q_{n-1}\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\Big) \\
&= \sqrt{\Delta_{2A}}^{\,n+1}\Big(\frac{G}{\sqrt{\Delta_{2A}}} P_n\big(\frac{G}{\sqrt{\Delta_{2A}}}\big) + \lambda_n P_{n-1}\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\Big)\frac{C'}{24} \\
&- \sqrt{\Delta_{2A}}^{\,n+2}\Big(\frac{G}{\sqrt{\Delta_{2A}}} Q_n\big(\frac{G}{\sqrt{\Delta_{2A}}}\big) + \lambda_n Q_{n-1}\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\Big) \\
&= \sqrt{\Delta_{2A}}^{\,n+1} P_{n+1}\big(\frac{G}{\sqrt{\Delta_{2A}}}\big)\frac{C'}{24} - \sqrt{\Delta_{2A}}^{\,n+2} Q_{n+1}\big(\frac{G}{\sqrt{\Delta_{2A}}}\big) \\
&= F_{k+4}.
\end{aligned}$$

Now we prove by induction that the $F_k(\tau)$ satisfies the equation $(\#')_k$. We can check the cases $k = 3$ and 7 directly. Assume F_{k-4} and F_k satisfy $(\#')_{k-4}$ and $(\#')_k$ respectively. Then by using (1) and the formulas

$$\vartheta(C) = -\frac{1}{4}G, \quad \vartheta(G) = -\frac{1}{2}C^3, \quad \vartheta(\Delta_{2A}) = 0$$

we have

$$\begin{aligned}
\vartheta^2(F_k) &= \vartheta\big(\vartheta(F_k)G - \frac{1}{2}C^3F_k\big) + \lambda_n\Delta_{2A}\vartheta^2(F_{k-4}) \\
&= \vartheta^2(F_k)G - \frac{1}{2}\vartheta(F_k)C^3 + \frac{3}{8}C^2GF_k - \frac{1}{2}C^3\vartheta(F_k) + \lambda_n\Delta_{2A}\vartheta^2(F_{k-4}) \\
&= \frac{k(k+2)}{64}C^2GF_k - C^3\vartheta(F_k) + \frac{3}{8}C^2GF_k + \frac{(k-4)(k-2)}{64}\lambda_n\Delta_{2A}C^2F_{k-4} \\
&= \frac{k^2+2k+24}{64}C^2GF_k + \frac{(k-4)(k-2)}{64}\lambda_n\Delta_{2A}C^2F_{k-4} - C^3\vartheta(F_k).
\end{aligned}$$

Hence we find

$$\begin{aligned}
&\vartheta^2(F_{k+4}) - \frac{(k+4)(k+6)}{64}C^2F_{k+4} \\
&= \Big(\frac{k^2+2k+24}{64} - \frac{(k+4)(k+6)}{64}\Big)C^2GF_k \\
&+ \Big(\frac{(k-4)(k-2)}{64} - \frac{(k+4)(k+6)}{64}\Big)\lambda_n\Delta_{2A}C^2F_{k-4} \\
&= -C^2\Big(\frac{k}{8}GF_k + C\vartheta(F_k) + \frac{k+1}{4}\lambda_n\Delta_{2A}F_{k-4}\Big).
\end{aligned}$$

The proof of the theorem therefore boils down to show the equation

$$\frac{k}{8}GF_k + C\vartheta(F_k) = -\frac{k+1}{4}\lambda_n \Delta_{2A} F_{k-4}.$$

For this we also proceed by induction. For $k = 7$ the equation is checked directly. Assuming that this is valid for k, we have

$$F_{k+4} = GF_k + \lambda_n \Delta_{2A} F_{k-4} = \frac{1}{2(k+1)}\big((k+2)GF_k - 8C\vartheta(F_k)\big)$$

and

$$\begin{aligned}
&\frac{k+4}{8}GF_{k+4} + C\vartheta(F_{k+4}) \\
&= \frac{k+4}{16(k+1)}G\big((k+2)GF_k - 8C\vartheta(F_k)\big) \\
&+ \frac{1}{2(k+1)}C\big(-\frac{1}{2}(k+2)C^3F_k + (k+2)G\vartheta(F_k) + 2G\vartheta(F_k) - 8C\vartheta^2(F_k)\big) \\
&= \frac{(k+2)(k+4)}{16(k+1)}(G^2 - C^4)F_k \\
&= -\frac{k+5}{4}\lambda_{n+1}\Delta_{2A}F_k.
\end{aligned}$$

Here we have used the (previous) induction assumption that F_k satisfies $(\#')_k$ and the relation $G^2 - C^4 = -256\Delta_{2A}$. This completes our proof.

References

1. J. Conway and S. Norton, *Monstrous Moonshine*, Bull. London Math. Soc. **11** (1979), 308–339.
2. M. Kaneko and M. Koike, *On modular forms arising from a differential equation of hypergeometric type, to appear in Ramanujan J.*
3. M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, "The Moduli Space of Curves", Progress in Math. **129** (1995), Birkhäuser, 165–172.
4. M. Kaneko and D. Zagier, *Supersingular j-invariants, Hypergeometric series, and Atkin's orthogonal polynomials*, AMS/IP Studies in Advanced Mathematics **7** (1998), 97–126.
5. M. Koike, *Modular forms on non-compact arithmetic triangle groups, preprint.*
6. M. Koike and T. Saijo, *in preparation.*
7. T. Saijo, *Modular forms and differential equations*, Kyushu University Master's thesis (1999.3).

Graduate School of Mathematics, Kyushu University 33, Fukuoka 812-8581, Japan
E-mail address: mkaneko@math.kyushu-u.ac.jp

Graduate School of Mathematics, Kyushu University, Ropponmatu, Fukuoka 810-8560, Japan
E-mail address: koike@math.kyushu-u.ac.jp